TRINITY COLLEGE

Year 11 Mathematics Specialist
Test 5 2016

Calculator Free
Matrices

STUDENT'S NAME ____________________

DATE: **TIME:** 50 minutes **MARKS**: ~~58~~ 55

INSTRUCTIONS:

Standard Items: Pens, pencils, ruler, eraser.

Special Items: Notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

1. (4 marks)

Determine matrices A and B given

- $A + B = \begin{bmatrix} 9 & -1 & 4 \\ 6 & -5\cdot5 & 3\cdot5 \end{bmatrix}$
- $a_{23} = b_{22} = 0\cdot5$
- $a_{21} = b_{12} = 2$
- $a_{11} = b_{11} - 1 = b_{13} + 1$

O - given

$$\underset{A}{\begin{bmatrix} 4 & -3 & 1 \\ 2 & -6 & 0.5 \end{bmatrix}} + \underset{B}{\begin{bmatrix} 5 & 2 & 3 \\ 4 & 0.5 & 3 \end{bmatrix}} = \begin{bmatrix} 9 & -1 & 4 \\ 6 & -5.5 & 3.5 \end{bmatrix}$$

$a_{11} + b_{11} = 9$

$b_{11} - 1 + b_{11} = 9$

$b_{11} = 5$

$4 = b_{13} + 1$

$3 = b_{13}$

2. (~~10~~ 6 marks)

(a) $A = \begin{bmatrix} 6k & k-7 \\ 3k & k+2 \end{bmatrix}$. Determine the value(s) of k such that A is singular. [3]

$6k(k+2) - 3k(k-7) = 0 \Rightarrow 3k^2 + 33k = 0$

$\Rightarrow 3k(k+11) = 0$

$\Rightarrow \underline{k = 0 \text{ or } k = -11}$

(b) If $A = \begin{bmatrix} x^2 & 3 \\ 1 & 3x \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 2 & x \end{bmatrix}$, determine the value(s) of x so that A and B are commutative for multiplication. i.e. $AB = BA$ [4]

$\begin{bmatrix} x^2 & 3 \\ 1 & 3x \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 2 & x \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & x \end{bmatrix}\begin{bmatrix} x^2 & 3 \\ 1 & 3x \end{bmatrix} \Rightarrow \begin{bmatrix} 3x^2+6 & 6x^2+3x \\ 3+6x & 6+3x^2 \end{bmatrix} = \begin{bmatrix} 3x^2+6 & 9+18x \\ 2x^2+x & 6+3x^2 \end{bmatrix}$

$\Rightarrow 3+6x = 2x^2 + x \Rightarrow 2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0$

$\Rightarrow x = -\frac{1}{2} \text{ or } 3$

$6x^2 + 3x = 9 + 18x$

$6x^2 - 15x - 9 = 0$

$2x^2 - 5x - 3 = 0 \Rightarrow \underline{x = -\frac{1}{2} \text{ or } 3}$

(c) Prove that $(PQ)^3 = I$, given that $QPQ = P^{-1}Q^{-1}P^{-1}$ and I is the identity matrix. [3]

$(PQ)^3 = PQPQPQ$

$= PQ\underline{P}P^{-1}Q^{-1}P^{-1}$

$= P\underline{Q}Q^{-1}P^{-1}$

$= PP^{-1}$

$= I$ QED

$QPQ = P^{-1}Q^{-1}P^{-1}$

$PQPQ = Q^{-1}P^{-1}$ $\times P$ (on left)

$PQPQP = Q^{-1}$ $\times P$ (on right)

$PQPQPQ = I$ as req'd $\times Q$ (" ")

ie $(PQ)^3 = I$

3. (8 marks)

Two matrices A and B are related by the equation $A + B = AB$.

(a) What does this equation imply about the dimensions of A and B? [2]

Same dimension and square

(b) (i) Use the equation given above to prove that $(I - A)(I - B) = I$ where I denotes an appropriate identity matrix. [3]

$$LHS = I^2 - IB - AI + AB$$
$$= I - B - A + AB$$
$$= I - B - A + A + B$$
$$= I = RHS.$$

(ii) Hence determine the inverse matrix $(I - A)^{-1}$ when $B = \begin{bmatrix} 8 & -8 & 5 \\ -4 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}$ [3]

$$(I-A)(I-B) = I$$
$$\Rightarrow (I-A)^{-1}(I-A)(I-B) = (I-A)^{-1}.I$$
$$I-B = (I-A)^{-1}$$
$$\Rightarrow \begin{bmatrix} -7 & 8 & -5 \\ 4 & -5 & 3 \\ -1 & 1 & -1 \end{bmatrix} = (I-A)^{-1}$$

8

4. (~~6~~ marks)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

There are only six possible matrices that can result from calculating A^n where $n = 1, 2, 3, 4, \ldots$

5

(a) Determine the six possible matrices [~~3~~]

$A^1 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $A^5 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

$A^6 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

(b) Show why there can be no more than these six. [1]

Since $A^6 = I$ then $A^7 = A$

$A^8 = A^2$ . . . . . etc.

(c) Using only this information, and showing working, determine A^{21}. [2]

$A^{21} = A^6 A^6 A^6 A^3 = I A^3$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

5. (7 marks)

The transformation T is formed by two successive applications of the transformation represented by $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\boxed{\begin{aligned} T &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M^2 \quad (1) \end{aligned}}$$

part (a)

(b) ~~(a)~~ Determine the image of the point (8, 0) under the transformation represented by M^{99} ~~[3]~~ [2]

$\therefore M^{99} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ since 99 is odd.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$ image is $(-8, 0)$

(c) ~~(b)~~ Determine the coordinates of the point whose image is (0, 10) under the transformation represented by M^{99}. [2]

$M^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\therefore$ coordinates are $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$

$= (0, -10)$

(d) ~~(c)~~ Comment on the geometrical effects of the transformation represented by M^{99}. [2]

rotation of 180° about (0,0)

6. (16 marks)

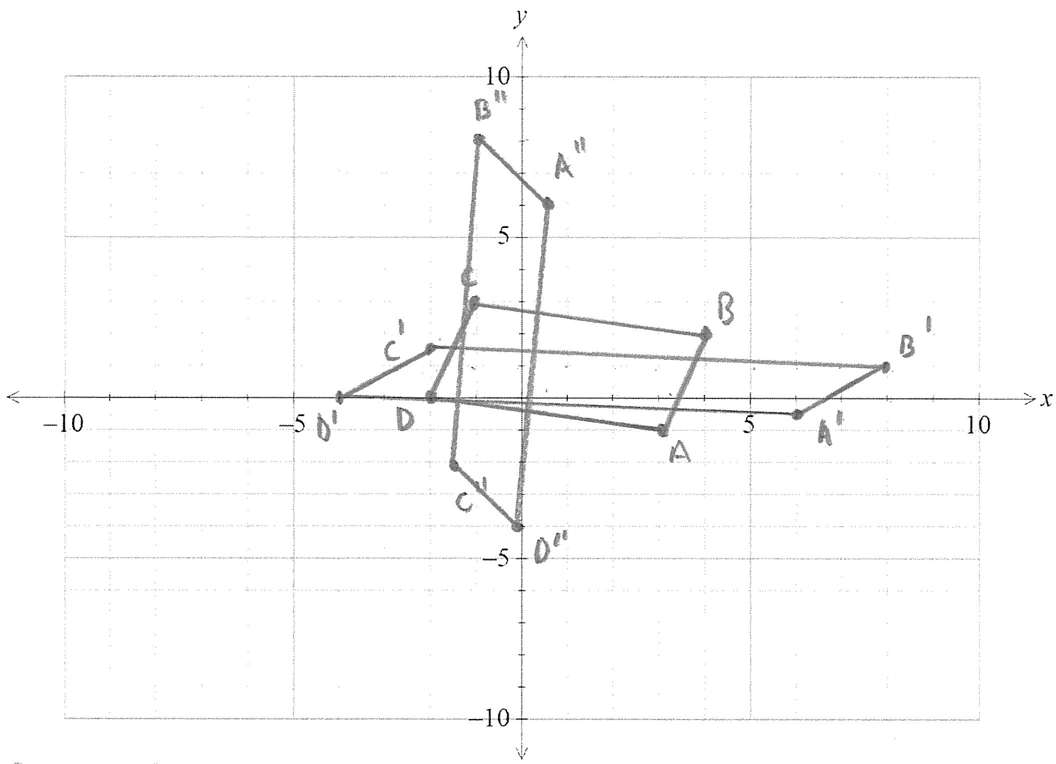
A parallelogram formed by the points A(3, –1), B(4, 2), C(–1, 3) and D(–2, 0) is transformed into A’B’C’D’ by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

(a) What are the coordinates of A’, B’, C’ and D’. [2]

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 3 & 4 & -1 & -2 \\ -1 & 2 & 3 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix} 6 & 8 & -2 & -4 \\ -\frac{1}{2} & 1 & \frac{3}{2} & 0 \end{bmatrix} \end{array}$$

$\therefore$ $A'(6, -\frac{1}{2})$, $B'(8, 1)$, $C'(-2, \frac{3}{2})$ $D'(-4, 0)$

(b) Draw ABCD and A’B’C’D’ on the axes below. [4]



(c) Is A’B’C’D’ a parallelogram? [1]

Yes, as A'B' and D'C' have the same gradient
+ B'C' and A'D' " " " "

(d) Compare the area of ABCD and A'B'C'D'. [1]

$\det\begin{bmatrix}2 & 0\\ 0 & \frac{1}{2}\end{bmatrix} = 1$ $\therefore$ Areas are the same

(e) Transform A'B'C'D' to A"B"C"D" using the matrix $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and draw the new quadrilateral on your axes. [3]

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}6 & 8 & -2 & -4\\ -\frac{1}{2} & 1 & \frac{3}{2} & 0\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -1 & -\frac{3}{2} & 0\\ 6 & 8 & -2 & -4\end{bmatrix}$$

$A''(\frac{1}{2}, 6)$ $B''(-1, 8)$ $C''(-\frac{3}{2}, -2)$ $D''(0, -4)$

(f) Describe, in words, the transformation of ABCD to A"B"C"D" . [2]

Dilation of factor 2 in x-dirn + $\frac{1}{2}$ in y dirn followed by a rotation of 90° abt (0,0)

(g) What single matrix would transform A"B"C"D" back to ABCD? [3]

ABCD $\rightarrow$ A"B"C"D" is $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & \frac{1}{2}\end{bmatrix}$

$= \begin{bmatrix}0 & -\frac{1}{2}\\ 2 & 0\end{bmatrix}$

inverse of $\begin{bmatrix}0 & -\frac{1}{2}\\ 2 & 0\end{bmatrix}$ is $\underline{\underline{\begin{bmatrix}0 & \frac{1}{2}\\ -2 & 0\end{bmatrix}}}$

7. (7 marks)

(a) Given that matrices A and B are commutative for multiplication, simplify the following expression. Justify your answer. [3]

$$A^2BA^{-1}B^{-1}$$

$= AABA^{-1}B^{-1}$

$= AB\underbrace{AA^{-1}}_{I}B^{-1}$ since A & B are commutative

$= A\underline{BB^{-1}}$

$= AI = \underline{\underline{A}}$

(b) Let W be an $n \times n$ non-singular matrix such that $6W^2 - 2W + I = 0$ where I is the identity matrix and O is the zero matrix.
Determine p and q such that $W^{-1} = pW^2 + qI$. [4]

$6W^2 - 2W + I = 0$

$\times W^{-1}$ $\quad 6WWW^{-1} - 2WW^{-1} + W^{-1} = 0$

ie $\quad 6W - 2I + W^{-1} = 0$

$\therefore W^{-1} = 2I - 6W$

$= 2I - \frac{6}{2}(6W^2 + I)$

$= 2I - 18W^2 - 3I$

$= -I - 18W^2$

$\therefore p = -18$

$q = -1$